Advanced Studies in Contemporary Mathematics
35 (2025), No. 4, pp. 369 - 386

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2025.35.4.009

First Kind Stirling Numbers and Some New Generating Functions

Amel Ghadri[1], Mourad Chelgham[2], Baghdadi Aloui[3]
[1] LMAM Laboratory and Department of Mathematics,
Mohamed Seddik Ben Yahia University, Jijel, Algeria
ghadriamel3@yahoo.com
[2] LMAM Laboratory and Department of Mathematics,
Mohamed Seddik Ben Yahia University, Jijel, Algeria
chelghamm@yahoo.fr
[3] Higher Institute of Industrial Systems of Gabes, Gabes University
Research Laboratory of Mathematics and Applications LR17ES11
Erriadh City, 6072 Zrig, Gabes, Tunisia
baghdadi.aloui@fsg.rnu.tn

Abstract

In this paper, we give some new generating functions of the products of Stirling numbers of the first kind with some known numbers and polynomials by making use of the symmetrizing endomorphism operators $\delta^k_{p_1p_2p_3}$ to the series $\sum\limits_{n=0}^{\infty} S_n(A)p_1^n z^n$.

2020 Mathematics Subject Classification: Primary 05E05; Secondary 11B39.

Keywords: Symmetric functions; Generating functions; First kind Stirling numabers; Generalized Tribonacci numbers.

1. Introduction

In 2019, the generalized Tribonacci number $\{W_n\}_{n\in\mathbb{N}}$ was introduced by Soykan, Okumus and Tasdemir [17]. These numbers are given by the following recurrence relation

$$W_n = aW_{n-1} + bW_{n-2} + cW_{n-3},\ n \geq 3,\ \text{and } (a,b,c) \in \mathbb{R}^3,$$

with initial values $W_0 = \alpha,\ W_1 = \beta,\ W_2 = \gamma$, where α, β and γ are arbitrary integers.
If we set $a = b = c = 1$ and $W_0 = 0, W_1 = 1, W_2 = 1$, then $\{W_n\}$ is the well-known Tribonacci numbers and if we take $a = b = c = 1$ and $W_0 = 3, W_1 = 1, W_2 = 3$, then we recover the Tribonacci-Lucas number.

There have been many studies on the generalized Tribonacci numbers, see for example [10, 9, 17]. The following are a few specific instances of the generalized Tribonacci sequence $W_n\{a,b,c;\alpha,\beta,\gamma\}$

Sequence	$W_n\{a,b,c;\alpha,\beta,\gamma\}$	Notation
Tribonacci	$W_n\{1,1,1;0,1,1\}$	T_n
Tribonacci-Lucas	$W_n\{1,1,1;3,1,3\}$	K_n
Padovan	$W_n\{0,1,1;1,1,1\}$	P_n
Perin	$W_n\{0,1,1;3,0,2\}$	R_n
Padovan-Perin	$W_n\{0,1,1;0,0,1\}$	S_n
Jacobsthal	$W_n\{1,1,2;0,1,1\}$	$J_n^{(3)}$
Jacobsthal-Lucas	$W_n\{1,1,2;2,1,5\}$	$j_n^{(3)}$

Table 1. Some special cases of the generzalized Tribonacci sequence.

Now, we introduce the generalized Tribonacci polynomials, denoted by $W_n(x)$, which defined by the recurrence relation [13]

$$W_n(x) = x^2W_{n-1}(x) + xW_{n-2}(x) + W_{n-3}(x),\ n \geq 3,$$

with initial conditions

$$\begin{aligned}
W_1(x) &= a,\\
W_2(x) &= b_2x^2 + b_1x + b_0,\\
W_3(x) &= c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,
\end{aligned}$$

where b_2, c_1 and c_4 are positive integers and others parameters are non negative integers.

The 2-orthogonal monic Chebyshev polynomials (2-classical) of the first kind $\{\widehat{T}_n\}_{n\geq 0}$ were examined in [11], and determined by the subsequent relationships:

$$\widehat{T}_n(x) = x\widehat{T}_{n-1}(x) - \alpha\widehat{T}_{n-2}(x) - \gamma\widehat{T}_{n-3}(x),\ n \geq 0,\ \gamma \neq 0,$$

with initial values $\widehat{T}_0(x) = 1,\ \widehat{T}_1(x) = x$ and $\widehat{T}_2(x) = x^2 - \alpha$, where α and γ are constants (see also [14]).

In 2013 by many researchers as it enabled them to retrieve numerous identities and renowned generating functions, utilizing the symmetric functions technique. For additional details check [3, 6, 7, 8, 15, 16]. In this contribution, we shall define a new useful operator denoted by $\delta^k_{p_1p_2p_2}$, for which we can develop, expand and demonstrate new findings based on our earlier ones.

The further contents of this paper are as follows: In Section 2, we introduce symmetric functions and some of its properties. We also give some more useful definitions which are used in the subsequent sections. In Section 3, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator. This main theorem unifies several previously known results about the generating functions. It is then used to find the new generating functions of products for some know numbers and polynomials, in Section 4 and Section 5.

2. Preliminaries and some properties

In this section, we give some definitions and properties of the symmetric functions that we will need in the sequel.

Definition 1. [1] *Let A and P be any two alphabets. We define $S_n(A-P)$ by the following form*

$$\frac{\Pi_{p\epsilon P}(1-pz)}{\Pi_{a\epsilon A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-P)z^n, \tag{2.1}$$

with the condition $S_n(A-P) = 0$ for $n < 0$.

Note that, Eq. (2.1) *can be rewritten in the following form*

$$\sum_{n=0}^{\infty} S_n(A-P)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)z^n\right),$$

Where

$$S_n(A-P) = \sum_{i=0}^{n} S_i(A)S_{n-i}(-P)z^n.$$

Remark 1. *Taking $A = 0$ in* (2.1), *this gives*

$$\sum_{n=0}^{\infty} S_n(-P)z^n = \prod_{p\epsilon P}(1-pz).$$

Definition 2. [5] *Let g be any function on $\mathbb{R}^n$. The divided difference operator is given by the following form*

$$\partial_{x_ix_{i+1}}g(x_1,\ldots,x_i,x_{i+1},\ldots,x_n)=\frac{g(x_1,\ldots,x_i,x_{i+1},\ldots,x_n)-g(x_1,\ldots,x_{i-1},x_{i+1},x_i,\ldots,x_n)}{x_i-x_{i+1}}.$$

Definition 3. [2] *Let n be positive integer and $A=\{a_1,a_2\}$ are set of given variables. Then, the $n-th$ symmetric function $S=(a_1+a_2)$ is defined by*

$$S_n(A)=S_n(a_1+a_2)=\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2},$$

with

$$\begin{aligned}S_0(A)&=S_0(a_1+a_2)=1,\\ S_1(A)&=S_1(a_1+a_2)=a_1+a_2,\\ S_2(A)&=S_2(a_1+a_2)=a_1^2+a_1a_2+a_2^2,\\ &\vdots\end{aligned}$$

Definition 4. [7] *The new symmetrizing $\delta^k_{e_1e_2}$is defined by*

$$\delta^k_{p_1p_2}f(p_1)=\frac{p_1^kf(p_1)-p_2^kf(p_2)}{p_1-p_2},\text{for all } k\in\mathbb{N}. \tag{2.2}$$

If $f(p_1)=p_1$, then Eq. (2.2) *can be written as follows*

$$\delta^k_{p_1p_2}f(p_1)=\frac{p_1^{k+1}-p_2^{k+1}}{p_1-p_2}=S_k(p_1+p_2).$$

3. Main results

In this section, we, firstly, start with the following new definitions and then the main theorems.

Definition 5. *The new symmetrizing $\delta^k_{p_1p_2p_3}$ is defined by*

$$\delta^k_{p_1p_2p_3}f(p_1)=\frac{p_2\delta^k_{p_1p_2}f(p_1)-p_3\delta^k_{p_1p_3}f(p_1)}{p_2-p_3}.$$

Definition 6. [12] *Let $P=\{p_1,p_2,p_3\}$ be an alphabet. Then we have*

$$S_n(P_3)=\frac{p_2S_n(p_1+p_2)-p_3S_n(p_1+p_3)}{p_2-p_3}.$$

Theorem 1. *Let $A=\{a_1,a_2,a_3\}$ and $P=\{p_1,p_2,p_3\}$ be two alphabets. The following result holds*

$$\sum_{n=0}^{\infty}S_n(A_3)S_{n+k-1}(P_3)z^n= \tag{3.1}$$

$$\frac{\begin{pmatrix} p_1^kp_3^k\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(p_1+p_3)p_3z^n\right)\\ -p_1^kp_2^k\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_3^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(p_1+p_2)p_2z^n\right)\end{pmatrix}}{(p_2-p_3)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_2^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)p_3^nz^n\right)},$$

for all $n,k\in\mathbb{N}_0$ and $k<n$.

Proof. By applying the operator $\delta^k_{p_1p_2p_3}$ to the sum $f(p_1z) = \sum\limits_{n=0}^{\infty} S_n(A)p_1^n z^n$, we obtain

$$\begin{aligned}
\delta^k_{p_1p_2p_3} f(p_1z) &= \delta^k_{p_1p_2p_3} \sum_{n=0}^{\infty} S_n(A)p_1^n z^n \\
&= \frac{p_2\delta^k_{p_1p_2} f(p_1z) - p_3\delta^k_{p_1p_3} f(p_1z)}{p_2 - p_3} \\
&= \frac{p_2\Big(\frac{p_1^k \sum_{n=0}^{\infty} S_n(A)p_1^n z^n - p_2^k \sum_{n=0}^{\infty} S_n(A)p_2^n z^n}{p_1-p_2}\Big) - p_3\Big(\frac{p_1^k \sum_{n=0}^{\infty} S_n(A)p_1^n z^n - p_3^k \sum_{n=0}^{\infty} S_n(A)p_3^n z^n}{p_1-p_3}\Big)}{p_2 - p_3} \\
&= \frac{p_2 \sum_{n=0}^{\infty} S_n(A)\frac{p_1^{n+k}-p_2^{n+k}}{p_1-p_2} z^n - p_3 \sum_{n=0}^{\infty} S_n(A)\frac{p_1^{n+k}-p_3^{n+k}}{p_1-p_3} z^n}{p_2 - p_3} \\
&= \sum_{n=0}^{\infty} S_n(A)\frac{p_2 S_{n+k-1}(p_1+p_2) - p_3 S_{n+k-1}(p_1+p_3)}{p_2-p_3} z^n \\
&= \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(P_3) z^n.
\end{aligned}$$

On the other hand, applying the $\delta^k_{p_1p_2p_3}$ to the sum $f(p_1z) = \dfrac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)p_1^n z^n}$, this gives

$$\begin{aligned}
\delta^k_{p_1p_2p_3} f(p_1z) &= \delta^k_{p_1p_2p_3} \frac{1}{\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n} \\
&= \frac{p_2\delta^k_{p_1p_2} \frac{1}{\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n} - p_3\delta^k_{p_1p_3} \frac{1}{\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n}}{p_2 - p_3} \\
&= \frac{p_2 \dfrac{\frac{p_1^k}{\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n} - \frac{p_2^k}{\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n}}{p_1 - p_2} - p_3 \dfrac{\frac{p_1^k}{\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n} - \frac{p_3^k}{\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n}}{p_1 - p_3}}{p_2 - p_3} \\
&= \frac{p_2}{p_2 - p_3}\left(\frac{p_1^k \sum_{n=0}^{\infty} S_n(-A)p_2^n z^n - p_2^k \sum_{n=0}^{\infty} S_n(-A)p_1^n z^n}{(p_1-p_2)(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n)(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n)}\right) \\
&- \frac{p_3}{p_2 - p_3}\left(\frac{p_1^k \sum_{n=0}^{\infty} S_n(-A)p_3^n z^n - p_3^k \sum_{n=0}^{\infty} S_n(-A)p_1^n z^n}{(p_1-p_3)(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n)(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n)}\right)
\end{aligned}$$

$$= \frac{p_2}{p_2-p_3}\left(\frac{-p_1^k p_2^k \sum\limits_{n=0}^{\infty} S_n(-A)\dfrac{p_1^{n-k}+p_2^{n-k}}{p_1+p_2}z^n}{(\sum\limits_{n=0}^{\infty} S_n(-A)p_1^n z^n)(\sum\limits_{n=0}^{\infty} S_n(-A)p_2^n z^n)}\right)$$

$$- \frac{p_3}{p_2-p_3}\left(\frac{-p_1^k p_3^k \sum\limits_{n=0}^{\infty} S_n(-A)\dfrac{p_1^{n-k}+p_3^{n-k}}{p_1+p_3}z^n}{(\sum\limits_{n=0}^{\infty} S_n(-A)p_1^n z^n)(\sum\limits_{n=0}^{\infty} S_n(-A)p_3^n z^n)}\right)$$

$$= \frac{\begin{pmatrix} p_1^k p_3^k \left(\sum\limits_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-k-1}(p_1+p_3)p_3 z^n\right) \\ -p_1^k p_2^k \left(\sum\limits_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-k-1}(p_1+p_2)p_2 z^n\right)\end{pmatrix}}{(p_2-p_3)\left(\sum\limits_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}.$$

This completes the proof. □

- For $A=\{a_1,a_2,a_3\}$, $P=\{p_1,p_2,p_3\}$ and $k\in\{0,1,2,3\}$ in the relation (3.1), we have the following results.

Theorem 2. [12] *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $P=\{p_1,p_2,p_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{\begin{bmatrix} 1-S_2(-A_3)S_2(-P_3)z^2+[S_3(-A_3)(S_2(-P_3)S_1(-P_3)-2S_3(-P_3)) \\ +S_1(-A_3)S_3(-P_3)]z^3-S_1(-A_3)S_1(-P_3)S_3(-A_3)S_3(-P_3)z^4 \\ +S_3^2(-A_3)S_3^2(-P_3)z^6\end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \quad (3.2)$$

From (3.2), *we get*

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_{n-1}(P_3)z^n = \frac{\begin{bmatrix} z-S_2(-A_3)S_2(-P_3)z^3+[S_3(-A_3)(S_2(-P_3)S_1(-P_3)-2S_3(-P_3)) \\ +S_1(-A_3)S_3(-P_3)]z^4-S_1(-A_3)S_1(-P_3)S_3(-A_3)S_3(-P_3)z^5 \\ +S_3^2(-A_3)S_3^2(-P_3)z^7\end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \quad (3.3)$$

From (3.3), *we deduce*

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)S_{n-2}(P_3)z^n = \frac{\begin{bmatrix} z^2-S_2(-A_3)S_2(-P_3)z^4+[S_3(-A_3)(S_2(-P_3)S_1(-P_3)-2S_3(-P_3)) \\ +S_1(-A_3)S_3(-P_3)]z^5-S_1(-A_3)S_1(-P_3)S_3(-A_3)S_3(-P_3)z^6 \\ +S_3^2(-A_3)S_3^2(-P_3)z^8\end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \quad (3.4)$$

Theorem 3. [12] *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $P=\{p_1,p_2,p_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{\begin{bmatrix} -S_1(-A_3)z+S_2(-A_3)S_1(-P_3)z^2-S_3(-A_3)[S_1^2(-P_3)-S_2(-P_3)]z^3 \\ +[S_1(-A_3)S_3(-A_3)S_3(-P_3)-S_2(-A_3)^2S_3(-P_3)]z^4 \\ +S_2(-A_3)S_3(-A_3)S_1(-P_3)S_3(-P_3)z^5 \\ -S_3^2(-A_3)S_2(-P_3)S_3(-P_3)z^6\end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}, \quad (3.5)$$

which gives

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_{n-2}(P_3)z^n = \frac{\begin{bmatrix} -S_1(-A_3)z^2 + S_2(-A_3)S_1(-P_3)z^2 - S_3(-A_3)[S_1^2(-P_3) - S_2(-P_3)]z^4 \\ +[S_1(-A_3)S_3(-A_3)S_3(-P_3) - S_2(-A_3)^2 S_3(-P_3)]z^5 \\ +S_2(-A_3)S_3(-A_3)S_1(-P_3)S_3(-P_3)z^6 \\ -S_3^2(-A_3)S_2(-P_3)S_3(-P_3)z^7 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \tag{3.6}$$

Theorem 4. [12] *Given two alphabets* $A = \{a_1, a_2, a_3\}$ *and* $P = \{p_1, p_2, p_3\}$ *, we have*

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n+1}(P_3)z^n = \frac{\begin{bmatrix} -S_1(-P_3) + S_1(-A_3)S_2(-P_3)z - S_3(-P_3)[S_1^2(-A_3) - S_2(-A_3)]z^2 \\ -S_3(-A_3)[S_2^2(-P_3) - S_1(-P_3)S_3(-P_3)]z^3 \\ +S_1(-A_3)S_3(-A_3)S_2(-P_3)S_3(-P_3)z^4 \\ -S_3^2(-P_3)S_2(-A_3)S_3(-A_3)z^5 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \tag{3.7}$$

Using (3.7)*, we obtain*

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_n(P_3)z^n = \frac{\begin{bmatrix} -S_1(-P_3)z + S_1(-A_3)S_2(-P_3)z^2 - S_3(-P_3)[S_1^2(-A_3) - S_2(-A_3)]z^3 \\ -S_3(-A_3)[S_2^2(-P_3) - S_1(-P_3)S_3(-P_3)]z^4 \\ +S_1(-A_3)S_3(-A_3)S_2(-P_3)S_3(-P_3)z^5 \\ -S_3^2(-P_3)S_2(-A_3)S_3(-A_3)z^6 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}, \tag{3.8}$$

which gives

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)S_{n-1}(P_3)z^n = \frac{\begin{bmatrix} -S_1(-P_3)z^2 + S_1(-A_3)S_2(-P_3)z^3 - S_3(-P_3)[S_1^2(-A_3) - S_2(-A_3)]z^4 \\ -S_3(-A_3)[S_2^2(-P_3) - S_1(-P_3)S_3(-P_3)]z^5 \\ +S_1(-A_3)S_3(-A_3)S_2(-P_3)S_3(-P_3)z^6 \\ -S_3^2(-P_3)S_2(-A_3)S_3(-A_3)z^7 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \tag{3.9}$$

Theorem 5. [12] *Given two alphabets* $A = \{a_1, a_2, a_3\}$ *and* $P = \{p_1, p_2, p_3\}$ *, we have*

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n+2}(P_3)z^n = \frac{\begin{bmatrix} (S_1^2(-P_3) - S_2(-P_3)) - S_1(-A_3)[S_1(-P_3)S_2(-P_3) - S_3(-P_3)]z \\ +[S_1^2(-A_3)S_1(-P_3)S_3(-P_3) + S_2(-A_3)(S_2^2(-P_3) - 2S_1(-P_3)S_3(-P_3))]z^2 \\ -S_2(-P_3)S_3(-P_3)[S_1(-A_3)S_2(-A_3) - S_3(-A_3)]z^3 \\ +S_3^2(-P_3)[S_2^2(-A_3) - S_1(-A_3)S_3(-A_3)]z^4 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}. \tag{3.10}$$

This becomes

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)S_n(P_3)z^n = \frac{\begin{bmatrix} (S_1^2(-P_3) - S_2(-P_3))z^2 - S_1(-A_3)[S_1(-P_3)S_2(-P_3) - S_3(-P_3)]z^3 \\ +[S_1^2(-A_3)S_1(-P_3)S_3(-P_3) + S_2(-A_3)(S_2^2(-P_3) - 2S_1(-P_3)S_3(-P_3))]z^4 \\ -S_2(-P_3)S_3(-P_3)[S_1(-A_3)S_2(-A_3) - S_3(-A_3)]z^5 \\ +S_3^2(-P_3)[S_2^2(-A_3) - S_1(-A_3)S_3(-A_3)]z^6 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}, \tag{3.11}$$

which, finally, gives

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{\begin{bmatrix} (S_1^2(-A_3) - S_2(-A_3))z^2 - S_1(-P_3)[S_1(-A_3)S_2(-A_3) - S_3(-A_3)]z^3 \\ +[S_1^2(-P_3)S_1(-A_3)S_3(-A_3) + S_2(-P_3)(S_2^2(-A_3) - 2S_1(-A_3)S_3(-A_3))]z^4 \\ -S_2(-A_3)S_3(-A_3)[S_1(-P_3)S_2(-P_3) - S_3(-P_3)]z^5 \\ +S_3^2(-A_3)[S_2^2(-P_3) - S_1(-P_3)S_3(-P_3)]z^6 \end{bmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right)}, \tag{3.12}$$

where

$$\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_3^n z^n\right) =$$

$$\begin{gathered} 1 - [S_1(-P_3)S_1(-A_3) + S_2(-P_3)S_1^2(-A_3)]z + [S_2(-A_3)S_1^2(-P_3) - 2S_2(-P_3)S_2(-A_3)]z^2 \\ -[S_1^3(-P_3)S_3(-A_3) + S_3(-P_3)(S_1^3(-A_3) - 6S_3(-A_3)) + (S_1(-A_3)S_2(-A_3) - 3S_3(-A_3)) \\ (S_1(-P_3)S_2(-P_3) - 3S_3(-P_3))]z^3 + [S_3(-P_3)S_1(-P_3)S_2(-A_3)S_1^2(-A_3) + (S_2^2(-A_3) - 2S_3(-A_3)S_1(-A_3)) \\ (S_2^2(-P_3) - 2S_2(-P_3)S_1(-P_3)) + S_3(-A_3)S_1(-A_3)(S_1^2(-P_3)S_2(-P_3) - 5S_3(-P_3)S_1(-P_3))]z^4 \\ -[S_3(-P_3)S_2(-P_3)(S_1(-A_3)S_2^2(-A_3) + 2S_3(-A_3)S_1(-A_3) - S_3(-A_3)S_2(-A_3)) \\ +S_3(-P_3)S_1^2(-P_3)(S_3(-A_3)S_1^2(-A_3) - 2S_3(-A_3)S_2(-A_3)) + S_1(-P_3)S_2(-P_3)S_2(-A_3)S_3(-A_3)]z^5 \\ +[S_3^2(-P_3)S_2^3(-A_3) + S_3(-P_3)S_3(-A_3)(S_1(-A_3)S_2(-A_3) - 3S_3(-A_3))(S_1(-P_3)S_2(-P_3) - 3S_3(-P_3)) \\ +S_3^2(-A_3)(S_2^3(-P_3) - 6S_3^2(-P_3))]z^6 - [S_3^2(-P_3)S_1(-P_3)S_3(-A_3)(S_2^2(-A_3) - S_3(-A_3)S_1(-A_3)) \\ +S_3^2(-A_3)S_1(-A_3)S_3(-P_3)S_2(-P_3)]z^7 + S_3^2(-P_3)S_2(-P_3)S_3^2(-A_3)S_2(-A_3)z^8 - S_3^3(-P_3)S_3^3(-A_3)z^9. \end{gathered}$$

4. Generating functions of the product of some known numbers and polynomials

In this part, we now derive the new generating functions of the product of some known numbers and polynomials.

4.1. Generating functions of the product of some known numbers. Firstly, using the substitutions

$$\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -1 \end{cases} \quad \textit{and} \begin{cases} S_1(-P_3) = -1 \\ S_2(-P_3) = -1 \\ S_3(-P_3) = -1 \end{cases},$$

in (3.2), (3.5) and (3.12), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1 - z^2 - 2z^3 - z^4 + z^6}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{z + z^2 + 2z^3 + z^5 - z^6}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}, \tag{4.2}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{z^2 + 2z^3 + 2z^4 - z^5}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}. \tag{4.3}$$

Than, we have the following theorem.

Theorem 6. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with Tribonacci-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} T_n K_n z^n = \frac{3 - 2z - 7z^2 - 12z^3 - 5z^4 - 5z^6}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}.$$

Proof. In [10], we have $T_n = S_n(A_3)$ and $K_n = 3S_n(P_3) - 2S_{n-1}(P_3) - S_{n-2}(P_3)$, then we see that

$$\begin{aligned}\sum_{n=0}^{\infty} T_n K_n z^n &= \sum_{n=0}^{\infty} S_n(A_3)(3S_n(P_3) - 2S_{n-1}(P_3) - S_{n-2}(P_3))z^n \\ &= 3\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n - 2\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n - \sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n.\end{aligned}$$

Using the relationships (4.1), (4.2) and (4.3), we obtain

$$\sum_{n=0}^{\infty} T_n K_n z^n = \frac{3 - 2z - 7z^2 - 12z^3 - 5z^4 - 5z^6}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9},$$

this completes the proof. □

Secondly, by substitute

$$\begin{cases} S_1(-A_3) = 0 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -1 \end{cases} \quad and \begin{cases} S_1(-P_3) = 0 \\ S_2(-P_3) = -1 \\ S_3(-P_3) = -1 \end{cases},$$

in (3.2), (3.4), (3.6), (3.8) and (3.12), we get

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1 - z^2 - 2z^3 + z^6}{1 - 2z^2 - 3z^3 - z^4 + z^5 + z^6 + z^8 - z^9}, \tag{4.4}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)S_{n-2}(P_3)z^n = \frac{z^2 - z^4 - 2z^5 + z^8}{1 - 2z^2 - 3z^3 - z^4 + z^5 + z^6 + z^8 - z^9}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_{n-2}(P_3)z^n = \frac{z^4 + 2z^5 - z^7}{1 - 2z^2 - 3z^3 - z^4 + z^5 + z^6 + z^8 - z^9}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_n(P_3)z^n = \frac{z^3 + z^4 - z^6}{1 - 2z^2 - 3z^3 - z^4 + z^5 + z^6 + z^8 - z^9}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{z^2 - z^4 - z^5}{1 - 2z^2 - 3z^3 - z^4 + z^5 + z^6 + z^8 - z^9}, \tag{4.8}$$

and we deduce the following theorem.

Theorem 7. *For $n \in \mathbb{N}$, the new generating function of the product of Padovan numbers with Perin numbers is given by*

$$\sum_{n=0}^{\infty} P_n R_n z^n = \frac{3 - 4z^2 - 3z^3 + 3z^4 + z^7}{1 - 2z^2 - 3z^3 - z^4 + z^5 + z^6 + z^8 - z^9}.$$

Proof. Recall that, we have (see [10]) $P_n = S_n(A_3) + S_{n-1}(A_3)$ and $R_n = 3S_n(P_3) - S_{n-2}(P_3)$, then we see that

$$\begin{aligned}\sum_{n=0}^{\infty} P_n R_n z^n &= \sum_{n=0}^{\infty} (S_n(A_3) + S_{n-1}(A_3))(3S_n(P_3) - S_{n-2}(P_3))z^n \\ &= 3\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n - \sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n \\ &\quad + 3\sum_{n=0}^{\infty} S_{n-1}(A_3)S_n(P_3)z^n - \sum_{n=0}^{\infty} S_{n-1}(A_3)S_{n-2}(P_3)z^n.\end{aligned}$$

By employing the connections (4.4), and (4.6)-(4.8), we obtain

$$\sum_{n=0}^{\infty} P_n R_n z^n = \frac{3-4z^2-3z^3+3z^4+z^7}{1-2z^2-3z^3-z^4+z^5+z^6+z^8-z^9},$$

hence, the desired result. $\square$

Theorem 8. *For $n \in \mathbb{N}$, the new generating function of the product of Padovan numbers with Padovan-Perin numbers is given by*

$$\sum_{n=0}^{\infty} P_n S_n z^n = \frac{z^2-z^7}{1-2z^2-3z^3-z^4+z^5+z^6+z^8-z^9}.$$

Proof. By referring to [10], we have $P_n = S_n(A_3) + S_{n-1}(A_3)$ and $S_n = S_{n-2}(P_3)$, this gives

$$\begin{aligned}\sum_{n=0}^{\infty} P_n S_n z^n &= \sum_{n=0}^{\infty} (S_n(A_3) + S_{n-1}(A_3)) S_{n-2}(P_3) z^n \\ &= \sum_{n=0}^{\infty} S_n(A_3) S_{n-2}(P_3) z^n + \sum_{n=0}^{\infty} S_{n-1}(A_3) S_{n-2}(P_3) z^n.\end{aligned}$$

Using the relationships (4.6) and (4.8), we obtain

$$\sum_{n=0}^{\infty} P_n S_n z^n = \frac{z^2-z^7}{1-2z^2-3z^3-z^4+z^5+z^6+z^8-z^9},$$

which completes the proof. $\square$

Theorem 9. *For $n \in \mathbb{N}$, the new generating function of the product of Perin numbers with Padovan-Perin numbers is given by*

$$\sum_{n=0}^{\infty} R_n S_n z^n = \frac{2z^2-2z^4-z^5-z^8}{1-2z^2-3z^3-z^4+z^5+z^6+z^8-z^9},$$

Proof. By the same method given in Theorem 3 and Theorem 4, the proof can be easily made. $\square$

Thirdly, setting

$$\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -2 \end{cases} \quad and \begin{cases} S_1(-P_3) = -1 \\ S_2(-P_3) = -1 \\ S_3(-P_3) = -2 \end{cases},$$

in (3.3), (3.5) and (3.9), this gives

$$\sum_{n=0}^{\infty} S_{n-1}(A_3) S_{n-1}(P_3) z^n = \frac{z-z^3-8z^4-4z^5+16z^7}{1-z-4z^2-29z^3-15z^4-16z^5+92z^6+32z^7+16z^8-64z^9}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} S_n(A_3) S_{n-1}(P_3) z^n = \frac{z+z^2+4z^3-2z^4+4z^5-8z^6}{1-z-4z^2-29z^3-15z^4-16z^5+92z^6+32z^7+16z^8-64z^9}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3) S_{n-1}(P_3) z^n = \frac{z^2+z^3+4z^4-2z^5+4z^6-8z^7}{1-z-4z^2-29z^3-15z^4-16z^5+92z^6+32z^7+16z^8-64z^9}, \tag{4.11}$$

and we deduce the following theorem.

Theorem 10. *For $n \in \mathbb{N}$, the new generating function of the product of third order Jacobsthal numbers with third order Jacobsthal-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} J_n^{(3)} j_n^{(3)} z^n = \frac{z+4z^2+11z^3+12z^4+8z^5-8z^6-32z^7}{1-z-4z^2-29z^3-15z^4-16z^5+92z^6+32z^7+16z^8-64z^9}.$$

Proof. By referred to [10], we have $J_n^{(3)} = S_{n-1}(A_3)$ and $j_n^{(3)} = 2S_n(P_3) - S_{n-1}(P_3) + 2S_{n-2}(P_3)$, then

$$\begin{aligned}\sum_{n=0}^{\infty} J_n^{(3)} j_n^{(3)} z^n &= \sum_{n=0}^{\infty}(2S_n(P_3) - S_{n-1}(P_3) + 2S_{n-2}(P_3))S_{n-1}(A_3)z^n \\ &= 2\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n - \sum_{n=0}^{\infty} S_{n-1}(A_3)S_{n-1}(P_3)z^n + 2\sum_{n=0}^{\infty} S_{n-2}(A_3)S_{n-1}(P_3)z^n.\end{aligned}$$

Using the relationships (4.9)-(4.11), we obtain

$$\sum_{n=0}^{\infty} J_n^{(3)} j_n^{(3)} z^n = \frac{2-z+4z^2+11z^3+12z^4+8z^5-8z^6-32z^7}{z-4z^2-29z^3-15z^4-16z^5+92z^6+32z^7+16z^8-64z^9},$$

which completes the proof of this theorem. □

4.2. **Generating functions of the product of some known polynomials.** Firstly, by substitution

$$\begin{cases} S_1(-A_3) = -x^2 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -1 \end{cases} \quad \textit{and} \begin{cases} S_1(-P_3) = -y^2 \\ S_2(-P_3) = -y \\ S_3(-P_3) = -1 \end{cases},$$

in (3.2), (3.5) and (3.12), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1-xyz^2+(x^2-y^2-2)z^3-x^2y^2z^4+z^6}{q_1(z)}, \tag{4.12}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{x^2z+xy^2z^2+y(1+y^3)z^3+xy^2z^5-yz^6}{q_1(z)}, \tag{4.13}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{(1+x^3)xz^2+(1+x^3)y^2z^3+(1+y^3)yx^2z^4-(1+y^3)xz^5}{q_1(z)}, \tag{4.14}$$

with

$$\begin{aligned} q_1(z) = 1 &- x^2y^2z - xy(x^3+y^3+2)z^2 - (x^6+y^6+x^3y^3+3x^3+3y^3+3)z^3 \\ &- x^2y^2(x^3+y^3+4)z^4 + xy(x^3-2x+1-y^2((x^3+2)y+1))z^5 \\ &+ (2y^3+2x^3+x^3y^3+3)z^6 + x^2y(y+1)z^7 + xyz^8 - z^9, \end{aligned}$$

and, then, the following theorem.

Theorem 11. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with Tribonacci-Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} T_n(x)K_n(y)z^n = \frac{\begin{bmatrix} 3-2x^2y^2z+xy(x^3-2y^3-4)z^2-(3x^2-6y^3+x^2y^3+6)z^3 \\ +(1-y^3)xyz^4+((1+y^3)xy+3)z^6 \end{bmatrix}}{q_1(z)}.$$

Proof. In [13], we have $T_n(x) = S_n(A_3)$ and $K_n(y) = 3S_n(P_3) - 2y^2 S_{n-1}(P_3) - yS_{n-2}(P_3)$. Then, we can easily see that

$$\sum_{n=0}^{\infty} T_n(x)K_n(y)z^n = \sum_{n=0}^{\infty} S_n(A_3)(3S_n(P_3) - 2y^2 S_{n-1}(P_3) - yS_{n-2}(P_3))z^n$$
$$= 3\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n - 2y^2 \sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n - y\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n.$$

According to relationships (4.12), (4.13) and (4.14), this gives the following equality

$$\sum_{n=0}^{\infty} T_n(x)K_n(y)z^n = \frac{\begin{bmatrix} 3 - 2x^2y^2z + xy(x^3 - 2y^3 - 4)z^2 - (3x^2 - 6y^3 + x^2y^3 + 6)z^3 \\ +(1 - y^3)xyz^4 + ((1 + y^3)xy + 3)z^6 \end{bmatrix}}{q_1(z)},$$

this completes the proof. □

Secondly, using the substitution

$$\begin{cases} S_1(-A_3) = -x^2 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -1 \end{cases} \textit{and} \begin{cases} S_1(-P_3) = -1 \\ S_2(-P_3) = -y \\ S_3(-P_3) = -y^2 \end{cases},$$

in (3.2), (3.8) and (3.11), we get

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1 - xyz^2 + (x^2y^2 + 2y^2 - y)z^3 - x^2y^2z^4 + y^4z^6}{q_2(z)}, \tag{4.15}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_n(P_3)z^n = \frac{z + x^2yz^2 + y^2(x^4 - x)z^3 + x^2y^3z^5 - xy^4z^6}{q - 2(z)}, \tag{4.16}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)S_n(P_3)z^n = \frac{(1 + y)z^2 + (y + y^2)x^2z^3 + (x^4y^2 + xy^2)z^4 - (1 + x^3)y^3z^5}{q_2(z)}, \tag{4.17}$$

where

$$\begin{aligned} q_2(z) = 1 &- x^2(1 - yx^2)z - (x + 2xy)z^2 - (1 + y^2(x^2 - 6) + y(x^2 + 3)(1 + 3y))z^3 \\ &+ (x^2y(x^3y - 3y - 1) - (x + y)^2)z^4 - (y^2(x^4 + 2x) - y^3(x^4 - 2x^2 + x) + xy)z^5 \\ &+ (y^3(x^2 + 3)(3y + 1) - y^3(6y + 1) - x^3y^4)z^6 + x^2y^3(y - 1)z^7 + xy^5z^8 - y^6z^9. \end{aligned}$$

Hence, the following result holds.

Proposition 1. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with Tricobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} T_n(x)J_n(y)z^n = \frac{1 - xyz^2 + (x^2y^2 - y(1 - 2y))z^3 - x^2y^2z^4 + y^4z^6}{q_2(z)}.$$

Corollary 1. *The following identity holds*

$$T_n(x)J_n(y) = S_n(A_3)S_n(P_3).$$

Theorem 12. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci-Lucas polynomials with Tricobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} K_n(x)J_n(y)z^n = \frac{\begin{bmatrix} 3 - 2x^2z - (4xy - 2x^4y - x)z^2 + (6y^2 - 3y - x^2y^2(2x^4 - 3) + x^3y^2(y - 1))z^3 \\ -(x^5y^2 + 4x^2y^2)z^4 + y^3(x - x^4)z^5 + (3y^4 + 2x^3y^4)z^6 \end{bmatrix}}{q_2(z)}.$$

Proof. By [13], we have $T_n(y) = S_n(P_3)$ and $J_n(x) = 3S_n(A_3) - 2x^2S_{n-1}(A_3) - xS_{n-2}(A_3)$, then we see that

$$\begin{aligned}\sum_{n=0}^{\infty} K_n(x)J_n(y)z^n &= \sum_{n=0}^{\infty}(3S_n(A_3) - 2x^2S_{n-1}(A_3) - xS_{n-2}(A_3))S_n(P_3)z^n \\ &= 3\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n - 2x^2\sum_{n=0}^{\infty} S_{n-1}(A_3)S_n(P_3)z^n - x\sum_{n=0}^{\infty} S_{n-2}(A_3)S_n(P_3)z^n.\end{aligned}$$

Using the relationships (4.15), (4.16) and (4.17), we obtain

$$\sum_{n=0}^{\infty} K_n(x)J_n(y)z^n = \frac{\begin{bmatrix}3 - 2x^2z - (4xy - 2x^4y - x)z^2 + (6y^2 - 3y - x^2y^2(2x^4 - 3) + x^3y^2(y-1))z^3 \\ -(x^5y^2 + 4x^2y^2)z^4 + y^3(x - x^4)z^5 + (3y^4 + 2x^3y^4)z^6\end{bmatrix}}{q_2(z)}.$$

Hence, the desired result. □

Now, the substitutions

$$\begin{cases} S_1(-A_3) = -x^2 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -1 \end{cases} \quad \textit{and} \begin{cases} S_1(-P_3) = -y \\ S_2(-P_3) = \alpha \\ S_3(-P_3) = -\gamma \end{cases},$$

in (3.2), (3.8) and (3.11), gives

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1 + \alpha xz^2 + (\gamma x^2 + \alpha y - 2\gamma)z^3 - \gamma yx^2z^4 + \gamma^2z^6}{q_3(z)}, \tag{4.18}$$

$$\sum_{n=0}^{\infty} S_{n-1}(A_3)S_n(P_3)z^n = \frac{yz - \alpha x^2z^2 + \gamma(x^4 - x)z^3 + (\alpha^2 - \gamma y)z^4 - \alpha\gamma x^2z^5 - \gamma^2xz^6}{q - 3(z)}, \tag{4.19}$$

$$\sum_{n=0}^{\infty} S_{n-2}(A_3)S_n(P_3)z^n = \frac{(y^2 - \alpha)z^2 + x^2(\gamma - \alpha y)z^3 + (\gamma yx^4 - x(\alpha^2 - 2\gamma y))z^4 + \alpha\gamma(x^3 + 1)z^5}{q_3(z)}, \tag{4.20}$$

with

$$\begin{aligned}q_3(z) = 1 &- (x^2y + \alpha x^4)z + (2x\alpha - xy^2)z^2 - (y^3 - \gamma(6 - x^6) + (x^3 + 3)(3\gamma - \alpha y))z^3 \\ &-(\alpha yx^5 + x^2(\alpha^2 - 2\gamma y - \alpha y^2) + 5xy)z^4 - (x^2(\alpha\gamma x^2 + 2) + \gamma y^2(x^4 + 2x) - x(\alpha y + 1))z^5 \\ &+(\gamma x^3(2\gamma - \alpha y) + \alpha(\alpha^2 - 3y) + 3\gamma(3 - 2\gamma))z^6 + x^2(\alpha\gamma - \gamma^2y)z^7 - \gamma^2\alpha xz^8 - \gamma^3z^9,\end{aligned}$$

and, then, we get the following results.

Proposition 2. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with 2-Chebyshev polynomials is given by*

$$\sum_{n=0}^{\infty} T_n(x)\widehat{T}_n(y)z^n = \frac{1 + \alpha xz^2 + (\gamma x^2 + \alpha y - 2\gamma)z^3 - \gamma yx^2z^4 + \gamma^2z^6}{q_3(z)}.$$

Corollary 2. *The following identity holds true*

$$T_n(x)\widehat{T}_n(y) = S_n(A_3)S_n(P_3).$$

Theorem 13. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci-Lucas polynomials with 2-Chebyshev polynomials is given by*

$$\sum_{n=0}^{\infty} K_n(x)\widehat{T}_n(y)z^n = \frac{\begin{bmatrix}3 - 2x^2yz + (\alpha(3 + 2x^3) - (y^2 - \alpha))xz^2 + (\gamma x^2(3\gamma + x - 2x^4) \\ +\alpha y(3 + x^3) - 6\gamma)z^3 + (\gamma xy(2 - 5x - x^4) + \alpha^2x(x - 2))z^4 \\ +\alpha\gamma x(x^3 - 1)z^5 + \gamma^2(3\gamma + 2x^3)z^6\end{bmatrix}}{q_3(z)}.$$

Proof. We prove the result by the same way given in Theorem 8. □

5. Generating functions of the product of Stirling numbers of first kind with some known numbers and polynomials

In this part, we now derive the new generating functions of the product of first kind Stirling numbers with some known numbers and polynomials.

Definition 7. *The first kind Stirling numbers, denoted by $s(n,k)$, is defined as a coefficients of the following*

$$(x)_n = x(x-1)(x-2)...(x-n+1) = \sum_{K=0}^{n} s(n,k)x^k.$$

Definition 8. [12] *Let $E_3 = \{1,2,3\}$ an alphabet, we have*

$$S_n(E_3) = \frac{3^{n+2}+1}{2} - 2^{n+2}.$$

5.1. Generating functions of the product of of first kind Stirling numbers with some known numbers.

Case 1. Taking,

$$\begin{cases} S_1(-A_3) = -6 \\ S_2(-A_3) = 11 \\ S_3(-A_3) = -6 \end{cases} \quad and \begin{cases} S_1(-P_3) = -1 \\ S_2(-P_3) = -1 \\ S_3(-P_3) = -1 \end{cases},$$

in Eqs. (3.2), (3.5) and (3.12), this gives

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1+11z^2+6z^3-36z^4+36z^6}{1+30z+33z^2-6z^3+131z^4+570z^5-73z^6-396z^8-216z^9}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{6z-11z^2+12z^3+85z^4-66z^5-36z^6}{1+30z+33z^2-6z^3+131z^4+570z^5-73z^6-396z^8-216z^9}, \tag{5.2}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{25z^2-602z^3-13z^4+132z^5}{1+30z+33z^2-6z^3+131z^4+570z^5-73z^6-396z^8-216z^9}, \tag{5.3}$$

then, we have the following proposition and theorem.

Proposition 3. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})T_n z^n = \frac{1+11z^2+6z^3-36z^4+36z^6}{1+30z+33z^2-6z^3+131z^4+570z^5-73z^6-396z^8-216z^9}.$$

Corollary 3. *The following identity holds*

$$(\frac{3^{n+2}+1}{2} - 2^{n+2})T_n = S_n(A_3)S_n(P_3).$$

Theorem 14. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Tribonacci-Lucas numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})K_n z^n = \frac{3-12z+30z^2+54z^3-265z^4+180z^6}{1+30z+33z^2-6z^3+131z^4+570z^5-73z^6-396z^8-216z^9}.$$

Proof. We have $S_n(A_3) = \frac{3^{n+2}+1}{2} - 2^{n+2}$ and $K_n = 3S_n(P_3) - 2S_{n-1}(P_3) - S_{n-2}(P_3)$. Then, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})K_n z^n &= \sum_{n=0}^{\infty} S_n(A_3)(3S_n(P_3) - 2S_{n-1}(P_3) - S_{n-2}(P_3))z^n \\ &= 3\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n - 2\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n - \sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n.\end{aligned}$$

Using the relationships (5.1), (5.2) and (5.3), we obtain

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})K_n z^n = \frac{3 - 12z + 30z^2 + 54z^3 - 265z^4 + 180z^6}{1 + 30z + 33z^2 - 6z^3 + 131z^4 + 570z^5 - 73z^6 - 396z^8 - 216z^9},$$

this completes the proof. □

Case 2. Making

$$\begin{cases} S_1(-A_3) = -6 \\ S_2(-A_3) = 11 \\ S_3(-A_3) = -6 \end{cases} \quad and \begin{cases} S_1(-P_3) = 0 \\ S_2(-P_3) = -1 \\ S_3(-P_3) = -1 \end{cases},$$

in (3.2), (3.5) and (3.12), we get

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1 + 11z^2 - 6z^3 + 36z^6}{1 + 36z + 22z^2 - 183z^3 + 49z^4 + 588z^5 + 215z^6 + 216z^7 - 396z^8 - 216z^9}, \quad (5.4)$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{6z + 6z^3 + 85z^4 - 36z^6}{1 + 36z + 22z^2 - 183z^3 + 49z^4 + 588z^5 + 215z^6 + 216z^7 - 396z^8 - 216z^9}, \quad (5.5)$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{25z^2 - 49z^4 + 66z^5}{1 + 36z + 22z^2 - 183z^3 + 49z^4 + 588z^5 + 215z^6 + 216z^7 - 396z^8 - 216z^9}. \quad (5.6)$$

Then, we have the following results.

Theorem 15. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Padovan numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})P_n z^n = \frac{1 + 6z + 11z^2 + 85z^4}{1 + 36z + 22z^2 - 183z^3 + 49z^4 + 588z^5 + 215z^6 + 216z^7 - 396z^8 - 216z^9}.$$

Proof. We have $S_n(A_3) = \frac{3^{n+2}+1}{2} - 2^{n+2}$ and $P_n = S_n(P_3) + S_{n-1}(P_3)$. Then, it is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})P_n z^n &= \sum_{n=0}^{\infty} S_n(A_3)(S_n(P_3) + S_{n-1}(P_3))z^n \\ &= \sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n + \sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n,\end{aligned}$$

Using the relationships (5.4) and (5.5), we obtain

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})P_n z^n = \frac{1 + 6z + 11z^2 + 85z^4}{1 + 36z + 22z^2 - 183z^3 + 49z^4 + 588z^5 + 215z^6 + 216z^7 - 396z^8 - 216z^9},$$

which completes the proof. □

Proposition 4. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Padovan-Perin numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})S_n z^n = \frac{25z^2-49z^4+66z^5}{1+36z+22z^2-183z^3+49z^4+588z^5+215z^6+216z^7-396z^8-216z^9}.$$

We can state the following corollary.

Corollary 4. *The following identity holds true*

$$(\frac{3^{n+2}+1}{2}-2^{n+2})S_n = S_n(A_3)S_{n-2}(P_3).$$

Theorem 16. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Perin numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})R_n z^n = \frac{3+18z+8z^2+304z^4-91z^5}{1+36z+22z^2-183z^3+49z^4+588z^5+215z^6+216z^7-396z^8-216z^9}.$$

Proof. By the same method given in theorem 11, the proof can be easily made. □

Case 3. Taking

$$\begin{cases} S_1(-A_3) = -6 \\ S_2(-A_3) = 11 \\ S_3(-A_3) = -6 \end{cases} \quad and \begin{cases} S_1(-P_3) = -1 \\ S_2(-P_3) = -1 \\ S_3(-P_3) = -2 \end{cases},$$

in (3.2), (3.5) and (3.12), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1+11z^2-18z^3-72z^4+144z^6}{w(z)}, \tag{5.7}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{6z-11z^2+12z^3+170z^4-132z^5-72z^6}{w(z)}, \tag{5.8}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{25z^2-60z^3-13z^4+198z^5}{w(z)}, \tag{5.9}$$

with

$$w(z) = 1+30z+33z^2-30z^3+249z^4-1074z^5+392z^6+1656z^7-1584z^8-1728z^9,$$

and we deduce the following proposition and theorem.

Proposition 5. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})J_n^{(3)} z^n = \frac{6z-11z^2+12z^3+170z^4-132z^5-72z^6}{w(z)}.$$

We can state the following corollary.

Corollary 5. *The following identity holds*

$$(\frac{3^{n+2}+1}{2}-2^{n+2})J_n^{(3)} = S_n(A_3)S_{n-1}(P_3).$$

Theorem 17. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Jacobsthal-Lucas numbers is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})j_n^{(3)} z^n = \frac{2-6z+83z^2-168z^3-340z^4+528z^5+360z^6}{w(z)},$$

Proof. By the same method given in Theorem 10, the proof can be easily made. □

5.2. **Generating functions of the product of first kind Stirling numbers with some known polynomials. Case 4.** The substitutions:

$$\begin{cases} S_1(-A_3) = -6 \\ S_2(-A_3) = 11 \\ S_3(-A_3) = -6 \end{cases} \quad and \begin{cases} S_1(-P_3) = -x^2 \\ S_2(-P_3) = -x \\ S_3(-P_3) = -1 \end{cases},$$

in (3.2), (3.5) and (3.12), we obtain

$$\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n = \frac{1 + 11xz^2 - 6(x^3+1)z^3 - 36x^2z^4 + 36z^6}{y_1(z)}, \tag{5.10}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n = \frac{6z - 11x^2z^2 + 6(x+x^4)z^3 + 85z^4 - 66x^2z^5 - 36xz^6}{y_1(z)}, \tag{5.11}$$

$$\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n = \frac{25z^2 - 60x^2z^3 + (36x^4 - 49x)z^4 - 66x^3z^5}{y_1(z)}, \tag{5.12}$$

with

$$\begin{aligned} y_1(z) = {} & 1 - (6x^2 - 36x)z + (11x^4 + 22x)z^2 + (48x^3 - 6x^2 - 36)z^3 + (167x^2 - 36x^5)z^4 \\ & +(588x + 66x^3 - 84x^4)z^5 + (251 - 324x^3)z^6 + 216(x + x^2)z^7 \\ & - 396xz^8 - 216z^9, \end{aligned}$$

then, we have the following proposition and theorem.

Proposition 6. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Tribonacci polynomials is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})T_n(x)z^n = \frac{1 + 11xz^2 - 6(x^3+1)z^3 - 36x^2z^4 + 36z^6}{y_1(z)},$$

We can state the following corollary.

Corollary 6. *The following identity holds*

$$(\frac{3^{n+2}+1}{2} - 2^{n+2})T_n(x) = S_n(A_3)S_n(P_3).$$

Theorem 18. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Tribonacci Lucas polynomials is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})K_n(x)z^n = \frac{\begin{bmatrix} 3 - 12x^2z + (22x^4 - 58x)z^2 - (12x^6 - 30x^3 + 18)z^3 \\ -x^2(229 + 36x^3)z^4 + 198x^4z^5 + (72x^3 + 108)z^6 \end{bmatrix}}{y_1(z)}.$$

Proof. By referring to [13], we have $K_n(x) = 3S_n(P_3) - 2x^2S_{n-1}(P_3) - xS_{n-2}(P_3)$ and $S_n(A_3) = \frac{3^{n+2}+1}{2} - 2^{n+2}$, then we see that

$$\begin{aligned} \sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2} - 2^{n+2})K_n(x)z^n &= \sum_{n=0}^{\infty} S_n(A_3)(3S_n(P_3) - 2x^2S_{n-1}(P_3) - xS_{n-2}(P_3))z^n \\ &= 3\sum_{n=0}^{\infty} S_n(A_3)S_n(P_3)z^n - 2x^2\sum_{n=0}^{\infty} S_n(A_3)S_{n-1}(P_3)z^n - x\sum_{n=0}^{\infty} S_n(A_3)S_{n-2}(P_3)z^n. \end{aligned}$$

Using the relationships (5.10), (5.11) and (5.12), we obtain

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})K_n(x)z^n = \frac{\begin{bmatrix}3-12x^2z+(22x^4-58x)z^2-(12x^6-30x^3+18)z^3\\ -x^2(229+36x^3)z^4+198x^4z^5+(72x^3+108)z^6\end{bmatrix}}{y_1(z)},$$

which completes the proof. □

Case 5. By substituting

$$\begin{cases}S_1(-A_3)=-6\\ S_2(-A_3)=11\\ S_3(-A_3)=-6\end{cases} \quad and \begin{cases}S_1(-P_3)=-1\\ S_2(-P_3)=-x\\ S_3(-P_3)=-x^2\end{cases},$$

in (3.2), we deduce the following proposition.

Proposition 7. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with Tricobsthal polynomials is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})J_n(x)z^n = \frac{1+11xz^2-(6x^2+6x)z^3-36x^2z^4+36x^4z^6}{y_2(z)},$$

with

$$\begin{aligned}y_2(z)=1&-(6-36x)z+(11+22x)z^2-(6-48x+36x^2)z^3+(167x^2-36x)z^4\\ &+(66x-84x^2+588x^3)z^5+(251x^2-324x^3)z^6-216(x^4-x^3)z^7-396x^5z^8-216x^6z^9.\end{aligned}$$

Corollary 7. *The following identity holds*

$$(\frac{3^{n+2}+1}{2}-2^{n+2})J_n(x)=S_n(A_3)S_n(P_3).$$

Case 6. The substitution

$$\begin{cases}S_1(-A_3)=-6\\ S_2(-A_3)=11\\ S_3(-A_3)=-6\end{cases} \quad and \begin{cases}S_1(-P_3)=-x\\ S_2(-P_3)=\alpha\\ S_3(-P_3)=-\gamma\end{cases},$$

in (3.2), gives the following proposition.

Proposition 8. *For $n \in \mathbb{N}$, the new generating function of the product of Stirling numbers of the first kind with 2-Chebyshev polynomials is given by*

$$\sum_{n=0}^{\infty}(\frac{3^{n+2}+1}{2}-2^{n+2})\widehat{T}_n(x)z^n = \frac{1-11\alpha z^2+6(\alpha x-\gamma)z^3-36\gamma xz^4-36\gamma^3z^6}{y_3(z)},$$

with

$$\begin{aligned}y_3(z)=1&-(6x+36\alpha)z+(11x^2-22\alpha)z^2-(36\gamma+6\gamma^3+48\alpha x)z^3\\ &+(167\gamma x+36\alpha x+49\alpha^2)z^4-(588\alpha\gamma+66\alpha x+84\gamma x^2)z^5\\ &+(251\gamma^2+36\alpha^3+288\alpha\gamma x)z^6-216(\gamma^2x+\alpha\gamma)z^7+396\alpha\gamma^2z^8-216\gamma^3z^9.\end{aligned}$$

Corollary 8. *The following identity holds*

$$(\frac{3^{n+2}+1}{2}-2^{n+2})\widehat{T}_n(x)=S_n(A_3)S_n(P_3).$$

6. Conclusion

Based on Theorem 1, we have derived some new generating functions of the first kind Stirling numbers with some known numbers and polynomials. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

Acknowledgments
The authors would like to thank the anonymous referees for their valuable comments and suggestions for improving the original version of this manuscript, we would like to extend développement our gratitude to Professor Ali BOUSSAYOUD for his invaluable ideas and assistance.

References

[1] A. Abderrezzak. Généralisation d'identités de Carlitz. Howard et Lehmer, Aequationes Math., **49** (1995), 36-46.
[2] K. Boubellouta, A. Boussayoud and M. Kerada, Symmetric Functions for Second-Order Recurrence Sequences, Tbil. Math. J. **13** (2020), 225-237.
[3] S. Boughaba and A. Boussayoud, On some identities and generating function of both k-Jacobsthal numbers and symmetric functions in several variables, Konualp J. Math. **7**, no. 2(2019), 235-242.
[4] S. Boughaba, A. Boussayoud, M. Kerada, construction of Symmetric Function of Generalized Fibonacci Numbers, Tamap Journal of Mathematics and statistics. **3** (2019), 1-7.
[5] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, Online.J. Anal. Comb. **12** (1) (2017), 1-10.
[6] A. Boussayoud, M. Chelgham, and S. Boughaba, On some identities and generating functions for Mersenne numbers and polynomials, Turk. J. Anal. Number Theory **6** (2018), 93-97.
[7] A. Boussayoud, M. Kerada, R. Sahali, Symmetrizing Operations on Some Orthogonal Polynomials, Int. Electron. J. Pure Appl. Math. 9, 191-199, (2015).
[8] A. Boussayoud, M. Kerada, and A. Abderrezzak, A generalization of some orthogonal polynomials, Proc. Math.Stat. **41** (2013), 229-235.
[9] M. Catalani, Identities for Tribonacci related sequences, https://arxiv.org/pdf/math/0209179.pdf, 2002.
[10] M. Chelgham, A. Boussayoud, Construction of symmetric functions of generalized Tribonacci numbers, J.Sci. Technol. Arts. **50** (2020), 65-74.
[11] K. Douak, P.|Maroni: On d-orthogonal Tchebychev polynomials. I, Appl. Numer Math. **24** (1997), 23-53.
[12] H. Merzouk, A. Boussayoud, A. Abderrezzak, Ordinary Generating functions of Binary products of Third-order recurrence relations and 2-orthogonal polynomials, Math. Slovaca **72**, (1) (2022), 11-34.
[13] H. Merzouk, A. Boussayoud, M. Chelgham, Generating functions of generalized Tribonacci and Tricobsthal polynomials, Montes Taurus J. Pure Appl. Math. **2** (2020), 7-37.
[14] T.A. Mesquita, A.Macedo, Chebyshev polynomials via quadratic and cubic decompositions of the canonical sequence, Integral Transforms Spec. Funct. **26** (12) (2015), 956-970.
[15] N. Saba, A. Boussayoud, On the bivariate Mersenne Lucas polynomials and their properties, Chaos Solitons Fractals. 146 (110899), 1-6 , (2021).
[16] N. Saba, A. Boussayoud, A. Abderrezzak, Complete homogeneous symmetric functions of third and second-order linear recurrence sequences, Electron. J. Math. Analysis Appl. 9, 226–242, (2021).
[17] Y. Soykan, I. Okumus, F. Tasdemir, On Generalized Tribonacci Sedenions, https://arxiv.org/pdf/1901.05312, 2019.